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Research Article

Mean Square Summability of Solution of Stochastic Difference Second-Kind Volterra Equation with Small Nonlinearity

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Stochastic difference second-kind Volterra equation with continuous time and small nonlinearity is considered. Via the general method of Lyapunov functionals construction, sufficient conditions for uniform mean square summability of solution of the considered equation are obtained.

1. Definitions and auxiliary results

Difference equations with continuous time are popular enough with researches [1–8]. Volterra equations are undoubtedly also very important for both theory and applications [3, 8–12]. Sufficient conditions for mean square summability of solutions of linear stochastic difference second-kind Volterra equations were obtained by authors in [10] (for difference equations with discrete time) and [8] (for difference equations with continuous time). Here the conditions from [8, 10] are generalized for nonlinear stochastic difference second-kind Volterra equations with continuous time. All results are obtained by general method of Lyapunov functionals construction proposed by Kolmanovskiĭ and Shaikhet [8, 13–21].

Let $\{\Omega,\mathfrak{F},\mathbf{P}\}$ be a probability space and let $\{\mathfrak{F}_t,\ t\geq t_0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, that is, $\mathfrak{F}_{t_1}\subset\mathfrak{F}_{t_2}$ for $t_1<t_2$, let H be a space of $\mathfrak{F}_t$-adapted functions x with values $x(t)$ in $\mathbb{R}^n$ for $t\geq t_0$ and the norm $\|x\|^2=\sup_{t\geq t_0}\mathbf{E}|x(t)|^2$.

Consider the stochastic difference second-kind Volterra equation with continuous time:

$$x(t+h_0)=\eta(t+h_0)+F(t,x(t),x(t-h_1),x(t-h_2),\dots),\quad t>t_0-h_0,\tag{1.1}$$

and the initial condition for this equation:

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta = \left[t_0 - h_0 - \max_{j\geq 1} h_j, t_0\right]. \tag{1.2}$$

Here $\eta \in H$, $h_0, h_1, \dots$ are positive constants, ϕ is an $\mathfrak{F}_{t_0}$-adapted function for $\theta \in \Theta$, such that $\|\phi\|_0^2 = \sup_{\theta\in\Theta} \mathbf{E}|\phi(\theta)|^2 < \infty$, the functional F with values in $\mathbb{R}^n$ satisfies the condition

$$\left|F(t, x_0, x_1, x_2, \dots)\right|^2 \leq \sum_{j=0}^{\infty} a_j \left|x_j\right|^2, \qquad A = \sum_{j=0}^{\infty} a_j < \infty. \tag{1.3}$$

A solution x of problem (1.1)-(1.2) is an $\mathfrak{F}_t$-adapted process $x(t) = x(t; t_0, \phi)$, which is equal to the initial function ϕ from (1.2) for $t \leq t_0$ and with probability 1 defined by (1.1) for $t > t_0$.

Definition 1.1. A function x from H is called

(i) uniformly mean square bounded if $\|x\|^2 < \infty$;

(ii) asymptotically mean square trivial if

$$\lim_{t\to\infty} \mathbf{E}\left|x(t)\right|^2 = 0; \tag{1.4}$$

(iii) asymptotically mean square quasitrivial if for each $t \geq t_0$,

$$\lim_{j\to\infty} \mathbf{E}\left|x(t + jh_0)\right|^2 = 0; \tag{1.5}$$

(iv) uniformly mean square summable if

$$\sup_{t\geq t_0} \sum_{j=0}^{\infty} \mathbf{E}\left|x(t + jh_0)\right|^2 < \infty; \tag{1.6}$$

(v) mean square integrable if

$$\int_{t_0}^{\infty} \mathbf{E}\left|x(t)\right|^2 dt < \infty. \tag{1.7}$$

Remark 1.2. It is easy to see that if the function x is uniformly mean square summable, then it is uniformly mean square bounded and asymptotically mean square quasitrivial.

Remark 1.3. It is evidently that condition (1.5) follows from (1.4), but the inverse statetent is not true.

Together with (1.1), we will consider the auxiliary difference equation

$$x(t+h_0) = F(t,x(t),x(t-h_1),x(t-h_2),\dots), \quad t > t_0 - h_0, \tag{1.8}$$

with initial condition (1.2) and the functional F, satisfying condition (1.3).

Definition 1.4. The trivial solution of (1.8) is called

(i) mean square stable if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon,t_0) > 0$ such that $\|x(t)\|^2 < \epsilon$ for all $t \geq t_0$ if $\|\phi\|_0^2 < \delta$;

(ii) asymptotically mean square stable if it is mean square stable and for each initial function ϕ, condition (1.4) holds;

(iii) asymptotically mean square quasistable if it is mean square stable and for each initial function ϕ and each $t \in [t_0,t_0+h_0)$, condition (1.5) holds.

Below some auxiliary results are cited from [8].

THEOREM 1.5. *Let the process η in (1.1) be uniformly mean square summable and there exist a nonnegative functional $V(t) = V(t,x(t),x(t-h_1),x(t-h_2),\dots)$, positive numbers c_1, c_2, and nonnegative function $\gamma : [t_0,\infty) \to \mathbb{R}$, such that*

$$\hat{\gamma} = \sup_{s\in[t_0,t_0+h_0)} \sum_{j=0}^{\infty} \gamma(s+jh_0) < \infty, \tag{1.9}$$

$$\mathbf{E}V(t) \leq c_1 \sup_{s\leq t} \mathbf{E}\left|x(s)\right|^2, \quad t \in [t_0,t_0+h_0), \tag{1.10}$$

$$\mathbf{E}\Delta V(t) \leq -c_2\mathbf{E}\left|x(t)\right|^2 + \gamma(t), \quad t \geq t_0, \tag{1.11}$$

where $\Delta V(t) = V(t+h_0) - V(t)$. Then the solution of (1.1)-(1.2) is uniformly mean square summable.

Remark 1.6. Replace condition (1.9) in Theorem 1.5 by condition

$$\int_{t_0}^{\infty} \gamma(t)dt < \infty. \tag{1.12}$$

Then the solution of (1.1) for each initial function (1.2) is mean square integrable.

Remark 1.7. If for (1.8) there exist a nonnegative functional $V(t) = V(t,x(t),x(t-h_1),x(t-h_2),\dots)$, and positive numbers c_1, c_2 such that conditions (1.10) and (1.11) (with $\gamma(t) \equiv 0$) hold, then the trivial solution of (1.8) is asymptotically mean square quasistable.

2. Nonlinear Volterra equation with small nonlinearity: conditions of mean square summability

Consider scalar nonlinear stochastic difference Volterra equation in the form

$$\begin{gathered} x(t+1) = \eta(t+1) + \sum_{j=0}^{[t]+r} a_j g(x(t-j)), \quad t > -1, \\ x(s) = \phi(s), \quad s \in [-(r+1),0]. \end{gathered} \tag{2.1}$$

Here $r \geq 0$ is a given integer, a_j are known constants, the process η is uniformly mean square summable, the function $g : \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$|g(x) - x| \leq \nu|x|, \quad \nu \geq 0. \tag{2.2}$$

Below in Theorems 2.1, 2.7, new sufficient conditions for uniform mean square summability of solution of (2.1) are obtained. Similar results for linear equations of type (2.1) were obtained by authors in [8, 10].

2.1. First summability condition. To get condition of mean square summability for (2.1), consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.3}$$

of dimension of $k+1$, $k \geq 0$, and the matrix equation

$$A'DA - D = -U, \tag{2.4}$$

with the solution D that is a symmetric matrix of dimension $k+1$ with the elements d_{ij}. Put also

$$\begin{gathered} \alpha_l = \sum_{j=l}^{\infty} |a_j|, \quad l = 0, \ldots, k+1, \qquad \beta_k = |a_k| + \sum_{m=0}^{k-1} \left| a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}} \right|, \\ A_k = \beta_k + \frac{1}{2}\alpha_{k+1}, \qquad S_k = d_{k+1,k+1}^{-1} - \alpha_{k+1}^2 - 2\beta_k \alpha_{k+1}. \end{gathered} \tag{2.5}$$

Theorem 2.1. *Suppose that for some $k \geq 0$, the solution D of (2.4) is a positive semidefinite symmetric matrix such that the condition $d_{k+1,k+1} > 0$ holds. If besides of that*

$$\alpha_{k+1}^2 + 2\beta_k \alpha_{k+1} < d_{k+1,k+1}^{-1}, \tag{2.6}$$

$$\nu < \frac{1}{\alpha_0}\left(\sqrt{A_k^2 + S_k} - A_k\right), \tag{2.7}$$

then the solution of (2.1) is uniformly mean square summable.

(For the proof of Theorem 2.1, see Appendix A.)

Remark 2.2. Condition (2.6) can be represented also in the form

$$\alpha_{k+1} < \sqrt{\beta_k^2 + d_{k+1,k+1}^{-1}} - \beta_k. \tag{2.8}$$

Remark 2.3. Suppose that in (2.1), $a_j = 0$ for $j > k$. Then $\alpha_{k+1} = 0$. So, if matrix equation (2.4) has a positive semidefinite solution D with $d_{k+1,k+1} > 0$ and ν is small enough to satisfy the inequality

$$\nu < \frac{1}{\alpha_0}\left(\sqrt{\beta_k^2 + d_{k+1,k+1}^{-1}} - \beta_k\right), \tag{2.9}$$

then the solution of (2.1) is uniformly mean square summable.

Remark 2.4. Suppose that the function g in (2.1) satisfies the condition

$$\left|g(x) - cx\right| \le \nu|x|, \tag{2.10}$$

where c is an arbitrary real number. Despite the fact that condition (2.10) is a more general one than (2.2), it can be used in Theorem 2.1 instead of (2.2). Really, if in (2.10) $c \neq 0$, then instead of a_j and g in (2.1), one can use $\hat{a}_j = a_j c$ and $\hat{g} = c^{-1}g$. The function $\hat{g}$ satisfies condition (2.2) with $\hat{\nu} = |c^{-1}|\nu$, that is, $|\hat{g}(x) - x| \le \hat{\nu}|x|$. In the case $c = 0$, the proof of Theorem 2.1 can be corrected by evident way (see Appendix A).

Remark 2.5. If inequalities (2.7), (2.8) hold and process η in (2.1) satisfies condition (1.12), then the solution of (2.1) is mean square integrable.

Remark 2.6. From Remark 1.7, it follows that if inequalities (2.7), (2.8) hold, then the trivial solution of (2.1) with $\eta(t) \equiv 0$ is asymptotically mean square quasistable.

2.2. Second summability condition. Put

$$\alpha = \sum_{j=1}^{\infty}\left|\sum_{m=0}^{\infty} a_m\right|, \qquad \beta = \sum_{j=0}^{\infty} a_j, \tag{2.11}$$

$$A = \alpha + \frac{1}{2}|\beta|, \qquad B = \alpha(|\beta| - \beta), \qquad S = (1-\beta)(1+\beta-2\alpha) > 0. \tag{2.12}$$

THEOREM 2.7. *Suppose that*

$$\beta^2 + 2\alpha(1-\beta) < 1, \tag{2.13}$$

$$\nu < \frac{1}{2|\beta|A}\left(\sqrt{(A+B)^2 + 2|\beta|AS} - (A+B)\right). \tag{2.14}$$

Then the solution of (2.1) is uniformly mean square summable.

(For the proof of Theorem 2.7, see Appendix B.)

Remark 2.8. Condition (2.13) can be written also in the form $|\beta| < 1$, $1 + \beta > 2\alpha$.

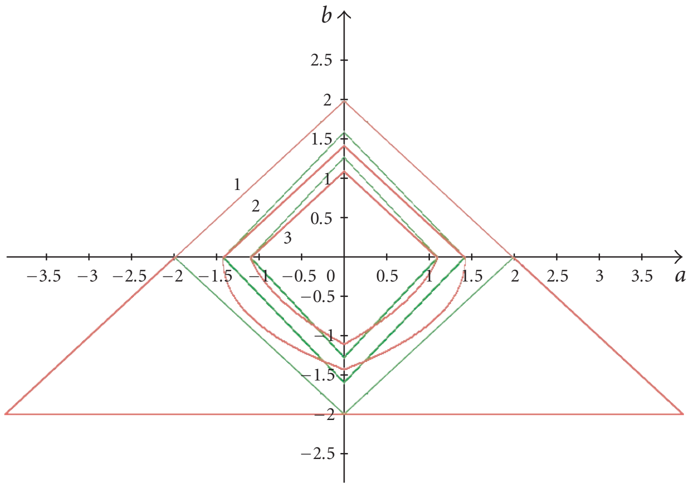

Figure 3.1. Regions of uniformly mean square summability for (3.1).

3. Examples

Example 3.1. Consider the difference equation

$$\begin{gathered} x(t+1)=\eta(t+1)+ag\big(x(t)\big)+bg\big(x(t-1)\big), \quad t>-1, \\ x(\theta)=\phi(\theta), \quad \theta\in[-2,0], \end{gathered} \tag{3.1}$$

with the function g defined as follows: $g(x)=c_1x+c_2\sin x$, $c_1\neq 0$, $c_2\neq 0$. It is easy to see that the function g satisfies condition (2.10) with $c=c_1$ and $\nu=|c_2|$. Via Remark 2.4 and (2.5), (2.6) for (3.1) in the case $k=0$, we have $\alpha_0=|c_1|(|a|+|b|)$, $\alpha_1=|c_1b|$, $\beta_0=|c_1a|$. Matrix equation (2.4) by the condition $|c_1a|<1$ gives $d_{11}^{-1}=1-c_1^2a^2>0$.

So, conditions (2.7), (2.8) via $\hat{\nu}=|c_1^{-1}c_2|$ take the form

$$|a|+|b|<\frac{1}{|c_1|}, \qquad |c_2|<|c_1|\frac{\sqrt{c_1^{-2}-|ab|-(3/4)b^2}-|a|-(1/2)|b|}{|a|+|b|}. \tag{3.2}$$

In the case $k=1$, we have $\alpha_0=|c_1|(|a|+|b|)$, $\alpha_1=|c_1b|$, $\alpha_2=0$. Besides (see [19]),

$$\beta_1=|c_1|\left(|b|+\frac{|a|}{1-c_1b}\right), \qquad d_{22}^{-1}=1-c_1^2b^2-c_1^2a^2\frac{1+c_1b}{1-c_1b} \tag{3.3}$$

and d_{22} is a positive one by the conditions $|c_1b|<1$, $|c_1a|<1-c_1b$.

Condition (2.8) trivially holds and condition (2.7) via $\hat{\nu}=|c_1^{-1}c_2|$ takes the form

$$|c_2|<\frac{(1-|c_1b|)(1-|c_1a|/1-c_1b)}{|a|+|b|}. \tag{3.4}$$

On Figure 3.1, the regions of uniformly mean square summability for (3.1) are shown, obtained by virtue of conditions (3.2) (the green curves) and (3.4) (the red curves) for

$c_1 = 0.5$ and different values of c_2: (1) $c_2 = 0$, (2) $c_2 = 0.2$, (3) $c_2 = 0.4$. On the figure, one can see that for $c_2 = 0$, condition (3.4) is better than (3.2) but for positive c_2, both conditions add to each other. Note also that for negative c_1, condition (3.4) gives a region that is symmetric about the axis a.

Example 3.2. Consider the difference equation

$$\begin{gathered} x(t+1) = \eta(t+1) + ag(x(t)) + \sum_{j=1}^{[t]+r} b^j g(x(t-j)), \quad t > -1, \\ x(\theta) = \phi(\theta), \quad \theta \in [-(r+1), 0],\ r \geq 0, \end{gathered} \tag{3.5}$$

with the function g that satisfies the condition $|g(x) - c_1 x| \leq c_2 |x|$, $c_1 \neq 0$, $c_2 > 0$.

In accordance with Remark 2.4, we will consider the parameters $c_1 a$ and $c_1 b^j$ instead of a and b^j. Via (2.11) by assumption $|b| < 1$, we obtain

$$\begin{gathered} \alpha = \sum_{j=1}^{\infty} \left| \sum_{m=j}^{\infty} c_1 b^m \right| = |c_1| \hat{\alpha}, \quad \hat{\alpha} = \frac{|b|}{(1-b)(1-|b|)}, \\ \beta = c_1 \hat{\beta}, \quad \hat{\beta} = a + \frac{b}{1-b}. \end{gathered} \tag{3.6}$$

Following (2.12), put also $A = |c_1|\hat{A}$, $\hat{A} = \hat{\alpha} + (1/2)|\hat{\beta}|$, $B = c_1^2 \hat{B}$, $\hat{B} = \hat{\alpha}\hat{\beta}(1 - \operatorname{sign}(\beta))$, $S = (1 - c_1\hat{\beta})(1 + c_1\hat{\beta} - 2|c_1|\hat{\alpha})$. Then condition (2.14) takes the form

$$c_2 < \frac{\sqrt{(\hat{A} + |c_1|\hat{B})^2 + 2|\hat{\beta}|\hat{A}S} - (\hat{A} + |c_1|\hat{B})}{2|\hat{\beta}|\hat{A}}. \tag{3.7}$$

To obtain another condition for uniformly mean square summability of the solution of (3.5), transform the sum from (3.5) for $t > 0$ in the following way:

$$\begin{aligned} \sum_{j=1}^{[t]+r} b^j g(x(t-j)) &= b \sum_{j=1}^{[t]+r} b^{j-1} g(x(t-j)) \\ &= b\left(g(x(t-1)) + \sum_{j=1}^{[t]-1+r} b^j g(x(t-1-j)) \right) \\ &= b\big[(1-a)g(x(t-1)) + x(t) - \eta(t)\big]. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.5), we transform (3.5) to the equivalent form

$$\begin{gathered} x(t+1) = \eta(t+1) + ag(\phi(t)) + \sum_{j=1}^{r-1} b^j g(\phi(t-j)), \quad t \in (-1, 0], \\ x(t+1) = \hat{\eta}(t+1) + ag(x(t)) + bx(t) + b(1-a)g(x(t-1)), \quad t > 0, \\ \hat{\eta}(t+1) = \eta(t+1) - b\eta(t). \end{gathered} \tag{3.9}$$

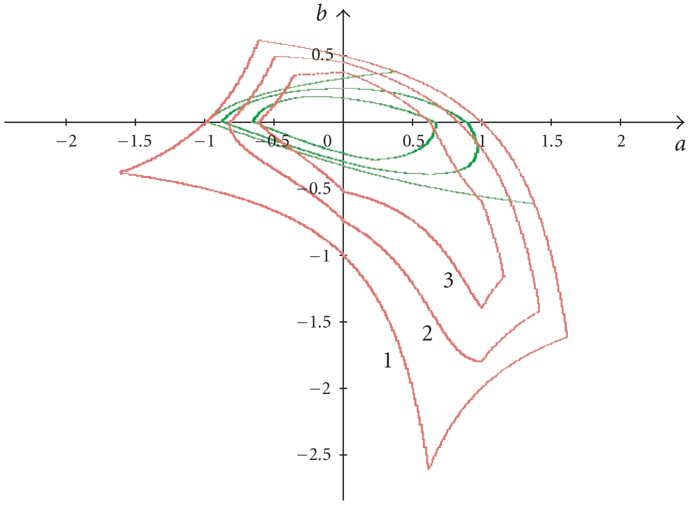

Figure 3.2. Regions of uniformly mean square summability given by conditions (3.7) and (3.10).

Using representation (3.9) of (3.5) without the assumption $|b| < 1$, one can show (see Appendix C) that by conditions $|c_1 b(1-a)| < 1$, $|c_1 a + b| < 1 - c_1 b(1-a)$ and

$$c_2 < \frac{(1 - |c_1 b(1-a)|)(1 - |c_1 a + b|/(1 - c_1 b(1-a)))}{|a| + |b(1-a)|}, \tag{3.10}$$

the solution of (3.5) is uniformly mean square summable.

Regions of uniformly mean square summability given by conditions (3.7) (the green curves), (3.10) (the red curves) are shown on Figure 3.2 for $c_1 = 1$ and different values of c_2: (1) $c_2 = 0$, (2) $c_2 = 0.2$, (3) $c_2 = 0.6$. On the figure, one can see that for $c_2 = 0$, condition (3.10) is better than (3.7), but for other values of c_2, both conditions add to each other. For negative c_1, condition (3.10) gives a region that is symmetric about the axis a.

Appendices

A. Proof of Theorem 2.1

In the linear case ($g(x) = x$), this result is obtained in [19]. So, here we will stress only the features of nonlinear case.

Suppose that for some $k \geq 0$, the solution D of (2.4) is a positive semidefinite symmetric matrix of dimension $k+1$ with the elements d_{ij} such that the condition $d_{k+1,k+1} > 0$ holds. Following the general method of Lyapunov functionals construction (GMLFC)

[8, 13–21] represents (2.1) in the form

$$x(t+1)=\eta(t+1)+F_1(t)+F_2(t), \tag{A.1}$$

where

$$F_1(t)=\sum_{j=0}^{k}a_jx(t-j),\qquad F_2(t)=\sum_{j=k+1}^{[t]+r}a_jx(t-j)+\sum_{j=0}^{[t]+r}a_j\big[g\big(x(t-j)\big)-x(t-j)\big]. \tag{A.2}$$

We will construct the Lyapunov functional V for (A.1) in the form $V=V_1+V_2$, where $V_1(t)=X'(t)DX(t)$, $X(t)=(x(t-k),\dots,x(t-1),x(t))'$.

Calculating and estimating $\mathbf{E}\Delta V_1(t)$ for (A.1) in the form $X(t+1)=AX(t)+B(t)$, where A is defined by (2.3), $B(t)=(0,\dots,0,b(t))'$, $b(t)=\eta(t+1)+F_2(t)$, similar to [19], one can show that

$$\begin{aligned}\mathbf{E}\Delta V_1(t)\le-\mathbf{E}x^2(t)+d_{k+1,k+1}\Bigg[&\big(1+\mu(1+\beta_k)\big)\mathbf{E}\eta^2(t+1)\\&+\big(\beta_k+\big(1+\mu^{-1}\big)\big(\nu\alpha_0+\alpha_{k+1}\big)\big)\sum_{j=0}^{[t]+r}f_{kj}^{\nu}\mathbf{E}x^2(t-j)\\&+\big(\mu^{-1}+\nu\alpha_0+\alpha_{k+1}\big)\sum_{m=0}^{k}\left|Q_{km}\right|\mathbf{E}x^2(t-m)\Bigg],\end{aligned} \tag{A.3}$$

where $\mu>0$,

$$\begin{gathered}f_{kj}^{\nu}=\begin{cases}\nu\left|a_j\right|, & 0\le j\le k,\\(1+\nu)\left|a_j\right|, & j>k,\end{cases}\\Q_{km}=a_m+\frac{d_{k-m,k+1}}{d_{k+1,k+1}},\quad m=0,\dots,k-1,\qquad Q_{kk}=a_k.\end{gathered} \tag{A.4}$$

Put now $\gamma(t)=d_{k+1,k+1}(1+\mu(1+\beta_k))\mathbf{E}\eta^2(t+1)$,

$$R_{km}=\begin{cases}\big(\mu^{-1}+\nu\alpha_0+\alpha_{k+1}\big)\left|Q_{km}\right|+\nu\big(\beta_k+\big(1+\mu^{-1}\big)\big(\nu\alpha_0+\alpha_{k+1}\big)\big)\left|a_m\right|, & 0\le m\le k,\\(1+\nu)\big(\beta_k+\big(1+\mu^{-1}\big)\big(\nu\alpha_0+\alpha_{k+1}\big)\big)\left|a_m\right|, & m>k.\end{cases} \tag{A.5}$$

Then (A.3) takes the form

$$\mathbf{E}\Delta V_1(t)\le-\mathbf{E}x^2(t)+\gamma(t)+d_{k+1,k+1}\sum_{m=0}^{[t]+r}R_{km}\mathbf{E}x^2(t-m). \tag{A.6}$$

Following GMLFC, choose the functional V_2 as follows:

$$V_2(t) = d_{k+1,k+1} \sum_{m=1}^{[t]+r} q_m x^2(t-m), \quad q_m = \sum_{j=m}^{\infty} R_{kj}, \; m = 0,1,\ldots, \tag{A.7}$$

and for the functional $V = V_1 + V_2$, we obtain

$$\mathbf{E}\Delta V(t) \leq -(1 - q_0 d_{k+1,k+1})\mathbf{E}x^2(t) + \gamma(t). \tag{A.8}$$

Since the process η is uniformly mean square summable, then the function γ satisfies condition (1.9). So if

$$q_0 d_{k+1,k+1} < 1, \tag{A.9}$$

then the functional V satisfies condition (1.11) of Theorem 1.5. It is easy to check that condition (1.10) holds too. So if condition (A.9) holds, then the solution of (2.1) is uniformly mean square summable.

Via (A.7), (A.5), (2.5), we have

$$q_0 = \alpha_{k+1}^2 + 2\beta_k\alpha_{k+1} + \nu^2\alpha_0^2 + (2\beta_k + \alpha_{k+1})\nu\alpha_0 + \mu^{-1}\left(\beta_k + (\nu\alpha_0 + \alpha_{k+1})^2\right). \tag{A.10}$$

Thus, if

$$\alpha_{k+1}^2 + 2\beta_k\alpha_{k+1} + \nu^2\alpha_0^2 + (2\beta_k + \alpha_{k+1})\nu\alpha_0 < d_{k+1,k+1}^{-1}, \tag{A.11}$$

then there exists a big $\mu > 0$ so that condition (A.9) holds, and therefore the solution of (2.1) is uniformly mean square summable. It is easy to see that (A.11) is equivalent to conditions of Theorem 2.1.

B. Proof of Theorem 2.7

Represent now (2.1) as follows:

$$x(t+1) = \eta(t+1) + F_1(t) + F_2(t) + \Delta F_3(t), \tag{B.1}$$

where $F_1(t) = \beta x(t)$, $F_2 = \beta(g(x) - x)$, β is defined by (2.11),

$$F_3(t) = -\sum_{m=1}^{[t]+r} B_m g(x(t-m)), \quad B_m = \sum_{j=m}^{\infty} a_j, \quad m = 0,1,\ldots. \tag{B.2}$$

Following GMLFC, we will construct the Lyapunov functional V for (2.1) in the form $V = V_1 + V_2$, where $V_1(t) = (x(t) - F_3(t))^2$. Calculating and estimating $\mathbf{E}\Delta V_1(t)$ via representation (B.1), similar to [8] we obtain

$$\begin{aligned} \mathbf{E}\Delta V_1(t) \leq{}& [1 + \mu(1+\nu)(\alpha + |\beta|)]\mathbf{E}\eta^2(t+1) + \lambda_\nu \sum_{m=1}^{[t]+r} |B_m| \mathbf{E}x^2(t-m) \\ &+ [\beta^2 - 1 + \alpha(1+\nu)(|\beta - 1| + (\nu + \mu^{-1})|\beta|) + \nu|\beta| + \nu^2\beta^2]\mathbf{E}x^2(t), \end{aligned} \tag{B.3}$$

where $\mu > 0$, α is defined by (2.11), $\lambda_\nu = (1+\nu)(|\beta - 1| + \nu|\beta| + \mu^{-1})$. Choosing V_2 in the form

$$V_2(t) = \lambda_\nu \sum_{m=1}^{[t]+r} \alpha_m x^2(t-m), \quad \alpha_m = \sum_{j=m}^{\infty} |B_j|, \quad m = 1,2,\dots, \tag{B.4}$$

for the functional $V = V_1 + V_2$, similar to [8] we have

$$\begin{aligned}\mathbf{E}\Delta V(t) \le {} & [1+\mu(1+\nu)(\alpha+|\beta|)]\mathbf{E}\eta^2(t+1) \\ & + [\beta^2 - 1 + 2\alpha(1+\nu)(|\beta-1| + \nu|\beta|) \\ & \quad + \nu|\beta| + \nu^2\beta^2 + \mu^{-1}\alpha(1+\nu)(1+|\beta|)]\mathbf{E}x^2(t).\end{aligned} \tag{B.5}$$

Thus, if

$$\beta^2 + 2\alpha(1+\nu)(|\beta-1| + \nu|\beta|) + \nu|\beta| + \nu^2\beta^2 < 1, \tag{B.6}$$

then there exists a big $\mu > 0$ so that the functional V satisfies the conditions of Theorem 1.5, and therefore, the solution of (2.1) is uniformly mean square summable. It is easy to check that (B.6) is equivalent to conditions of Theorem 2.7.

C. Proof of condition (3.10)

Following GMLFC, represent (3.9) in the form

$$x(t+1) = \hat{\eta}(t+1) + \hat{F}_1(t) + \hat{F}_2(t), \tag{C.1}$$

where $\hat{F}_1(t) = \hat{a}_0 x(t) + \hat{a}_1 x(t-1)$, $\hat{F}_2(t) = a_0\hat{g}(x(t)) + a_1\hat{g}(x(t-1))$, $a_0 = a$, $a_1 = b(1-a)$, $\hat{a}_0 = c_1 a + b$, $\hat{a}_1 = c_1 a_1$, $\hat{g}(x) = g(x) - c_1 x$. Using system (C.1) as $X(t+1) = \hat{A}X(t) + \hat{B}(t)$, where

$$X(t) = \begin{pmatrix} x(t-1) \\ x(t) \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & 1 \\ \hat{a}_1 & \hat{a}_0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ \hat{\eta}(t+1) + \hat{F}_2(t) \end{pmatrix}, \tag{C.2}$$

one has to repeat the proof of Theorem 2.1. Equation (2.4) with the matrix $A = \hat{A}$ by the conditions $|\hat{a}_1| < 1$, $|\hat{a}_0| < 1 - \hat{a}_1$ has a positive semidefinite solution $\hat{D}$ such that

$$\hat{d}_{22}^{-1} = 1 - \hat{a}_1^2 - \hat{a}_0^2 \frac{1+\hat{a}_1}{1-\hat{a}_1} > 0. \tag{C.3}$$

Since for (3.9) $\alpha_2 = 0$, then similar to (A.11) we obtain $c_2^2\alpha_0^2 + 2\hat{\beta}_1 c_2 \alpha_0 < \hat{d}_{22}^{-1}$, where

$$\alpha_0 = |a_0| + |a_1| = |a| + |b(1-a)|, \qquad \hat{\beta}_1 = |\hat{a}_1| + \frac{|\hat{a}_0|}{1-\hat{a}_1} = |c_1 b(1-a)| + \frac{|c_1 a + b|}{c_1 b(1-a)}. \tag{C.4}$$

Via (2.9) and Remark 2.3, this condition is equivalent to (3.10).

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